

Applications of Two Body Dirac Equations to Hadron and Positronium Spectroscopy

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Abstract

We review recent applications of the Two Body Dirac equations of constraint dynamics to meson spectroscopy and describe new extensions to three-body problems in their use in the study of baryon spectroscopy. We outline unique aspects of these equations for QED bound states that distinguish them among the various other approaches to the relativistic two body problem. Finally we discuss recent theoretical solutions of new peculiar bound states for positronium arising from the Two Body Dirac equations of constraint dynamics, assuming point particles for the electron and the positron.

1 Introduction

The Two-Body Dirac equations (TBDE) of Constraint Dynamics have dual origins. On the one hand they arise as one of the many quasipotential reductions of the Beth Salpeter equation (BSE)[1]. On the other they arise independently from the development of a consistent covariant approach to the two-body problem in relativistic classical mechanics independent of QFT[2]. In this talk we describe these two aspects and then go on to discuss applications to hadron spectroscopy[3],[4],[5]. The last part of our talk explains the importance we put on numerical QED tests [6] of the TBDE and some speculative theoretical results concerning new positronium states[7].

2 Quasipotential Reduction of the Bethe-Salpeter Equation

Two body Bethe-Salpeter equation [8] for spin-zero bound states is¹

$$G_0^{-1}\Psi = (p_1^2 + m_1^2)(p_2^2 + m_2^2)\Psi = K\Psi.$$

The irreducible kernel K is obtained from the off-mass-shell scattering amplitude T

$$T = K + KG_0T,$$

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¹The irreducible Bethe-Salpeter kernel K would in general contain charge renormalization and vacuum polarization graphs and could contain self-energy terms transferred from the inverse propagators.

and would in general contain charge renormalization and vacuum polarization graphs and could contain self-energy terms transferred from the inverse propagators.

The problems of the two body Bethe-Salpeter equation are its technical complexity and the existence of abnormal solutions excitations in the relative time-energy with no proper nonrelativistic limit [9], [10]. Recent work with static models has indicated, however, that these abnormal solutions disappear if one includes all ladder and cross ladder diagrams [11]. This supports Wick's conjecture on defects of ladder approximations. In the mean time numerous 3D quasipotential reductions of the Bethe-Salpeter equation had been proposed. In fact, they can be, in principle, infinite in number[12].

Reductions of the BSE can be obtained from iterating the Bethe Salpeter equation around a three-dimensional Lorentz invariant hypersurface in relative momentum (p) space. This leads to invariant three-dimensional wave equations for relative motion. The resultant 3D wave equation is not unique, but depends on the nature of the 3D hypersurface. We choose Todorov's quasipotential equation [13] which has this Schrödinger-like form

$$(p^2 + \Phi(x_1 - x_2)) \psi = b^2(w) \psi, \quad (1)$$

The 3D hypersurface restriction on the relative momentum $p = (p_1 - p_2)/2$ ($m_1 = m_2$) is defined by

$$\begin{aligned} p \cdot P \psi &= 0, \\ P &= p_1 + p_2. \end{aligned} \quad (2)$$

This eliminates from the start the problems associated with relative time/energy.

Defining

$$p_{\perp} = p + \hat{P} \cdot p \hat{P}, \quad p_{\perp} \cdot \hat{P} = 0, \quad \hat{P} = \frac{P}{\sqrt{-P^2}}, \quad \hat{P}^2 = -1,$$

we have

$$p^2 \psi = p_{\perp}^2 \psi.$$

The effective eigenvalue in the Schrödinger-like equation is

$$b^2 = \frac{1}{4w^2} [w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2],$$

with w the c.m. invariant energy

$$w = \sqrt{-P^2}.$$

The quasipotential Φ in Eq. (1) is related to scattering amplitude by a Lippmann Schwinger type equation

$$T - \Phi - \Phi \frac{1}{p_{\perp}^2 - b^2 - i0} T = 0. \quad (3)$$

The elastic unitarity condition[13],

$$T - T^{\dagger} = \pi i T \delta(p_{\perp}^2 - b^2) T^{\dagger},$$

leads to arbitrariness in the Green function

$$\frac{1}{p_{\perp}^2 - b^2 - i0} f(p_{\perp}^2), \quad f(b^2) = 1$$

and to the multiplicity of 3D reductions of BSE. Todorov's choice is $f(p_\perp^2) = 1$, and his equation displays exact relativistic two-body kinematics in the absence of interactions,

$$w = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}.$$

The restriction of $p \cdot P\psi = 0$ on the time-like component of the relative momentum is compatible with Eq. (1) provided

$$[p \cdot P, \Phi] \psi = 0.$$

This forces Φ to depend on $x_1 - x_2$ only through the transverse component,

$$x_\perp^\mu = \left(\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu \right) (x_1 - x_2)_\nu, \quad x_\perp \cdot \hat{P} = 0.$$

Thus, in the c.m. frame, the hypersurface restriction $p \cdot P\psi = 0$ not only eliminates the relative energy ($p\psi = (0, \mathbf{p})\psi$) but implies that the relative time does not appear, i.e. $(x_\perp = (0, \mathbf{r}))$.

The formal solution of Eq. (3) is

$$\Phi = T \left(1 + \frac{1}{p_\perp^2 - b^2 - i0} T \right)^{-1}.$$

A nonperturbative approximate solution to this equation has been obtained [14] for this for both world scalar and vector interactions which a) includes all ladder and cross ladder diagrams for $T = \sum_{n=1}^{\infty} T^{(n)}$ and b) includes iterations that result from the geometric series expansion

$$\left(1 + \frac{1}{p_\perp^2 - b^2 - i0} T \right)^{-1} = \sum_{m=1}^{\infty} (-)^m \left(\frac{1}{p_\perp^2 - b^2 - i0} T \right)^m.$$

The iterations are called Constraint Diagrams. For QED-like field theories, [14] uses a scheme that adapts Eikonal approximation for ladder, cross Ladder, and constraint diagrams to bound states. Applied through all orders it gives for scalar exchange the quasipotential

$$\Phi = 2m_w S + S^2,$$

while for vector exchange

$$\Phi = 2\varepsilon_w A - A^2.$$

The kinematical variables

$$\begin{aligned} m_w &= \frac{m_1 m_2}{w}, \\ \varepsilon_w &= \frac{w^2 - m_1^2 - m_2^2}{2w}, \end{aligned}$$

satisfy Einstein relation

$$b^2 = \varepsilon_w^2 - m_w^2,$$

and corresponds to the energy and reduced mass for the fictitious particle of relative motion. The effects of ladder and cross ladder diagrams thus embedded in their c.m. energy dependencies

By way of the minimal substitutions

$$\begin{aligned}\varepsilon_w &\rightarrow \varepsilon_w - A, \\ m_w &\rightarrow m_w + S,\end{aligned}$$

one can modify the free two-body equation

$$p^2\psi = (\varepsilon_w^2 - m_w^2)\psi = b^2\psi,$$

to

$$(p^2 + 2\varepsilon_w A - A^2 + 2m_w S + S^2)\psi = (\varepsilon_w^2 - m_w^2)\psi = b^2\psi,$$

in which the two particles interact by way of scalar and vector potentials. The form of

$$\Phi = 2m_w S + S^2 + 2\varepsilon_w A - A^2$$

is valid for more general potentials than the invariant Coulomb forms

$$-\frac{\alpha}{r} = -\frac{\alpha}{\sqrt{x_\perp^2}},$$

for which they are derived.

3 Two Body Dirac Equations of Constraint Dynamics

The Two-Body Dirac equations provide a manifestly covariant 3D reduction of the BSE for two spin-1/2 particles[2]. Furthermore, the constraint approach [16] provides a route around the Currie-Jordan-Sudarshan “non-interaction theorem” [15] which apparently forbade canonical 4-dimensional treatment of the relativistic N - body problem. As with the 3D quasipotential equation, the TBDE covariantly eliminates relative time and energy. For two particles interacting through scalar and vector interactions the TBDE are given by

$$\begin{aligned}\mathcal{S}_1\psi &\equiv \gamma_{51}(\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1 + \tilde{S}_1)\psi = 0, \\ \mathcal{S}_2\psi &\equiv \gamma_{52}(\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2 + \tilde{S}_2)\psi = 0,\end{aligned}$$

in which ψ is a 16 component spinor. The operators are compatible with

$$[\mathcal{S}_1, \mathcal{S}_2]\psi = 0, \text{ implying } \tilde{A}_i = \tilde{A}_i(x_\perp), \tilde{S}_i = \tilde{S}_i(x_\perp).$$

One can see the connection to the spin 0 quasipotential results using $\varepsilon_1, \varepsilon_2$, the c.m. particle energies

$$\begin{aligned}\varepsilon_1 + \varepsilon_2 &= w, \quad \varepsilon_1 - \varepsilon_2 = \frac{m_1^2 - m_2^2}{w}, \\ \varepsilon_1 &= \frac{1}{2} \left(w + \frac{(m_1^2 - m_2^2)}{w} \right), \quad \varepsilon_2 = \frac{1}{2} \left(w + \frac{(m_2^2 - m_1^2)}{w} \right).\end{aligned}$$

Using

$$\begin{aligned} p_1 &= \varepsilon_1 \hat{P} + p, \quad p_2 = \varepsilon_2 \hat{P} - p, \\ p &\equiv \frac{\varepsilon_2 p_1 - \varepsilon_1 p_2}{w}, \end{aligned}$$

we rewrite $p \cdot P\psi = 0$ and $(p^2 + \Phi)\psi = b^2(w)\psi$ as[16]

$$\begin{aligned} \mathcal{H}_1\psi &= (p_1^2 + m_1^2 + \Phi)\psi = 0, \\ \mathcal{H}_2\psi &= (p_2^2 + m_2^2 + \Phi)\psi = 0. \end{aligned}$$

The compatibility condition

$$[\mathcal{H}_1, \mathcal{H}_2]\psi = 0,$$

is satisfied provided that

$$\Phi = \Phi(x_\perp).$$

For the TBDE, $[\mathcal{S}_1, \mathcal{S}_2]\psi = 0$ also restricts the spin dependence of \tilde{A}_i^μ , \tilde{S}_i by determining their dependence on γ_1, γ_2

$$\tilde{A}_i^\mu = \tilde{A}_i^\mu(A(r), V(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2), \quad \tilde{S}_i = \tilde{S}_i(S(r), A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2).$$

with vector interactions \tilde{A}_i^μ depending on electromagnetic $A(r)$ time-like vector $V(r)$ invariant interactions through respective vertex forms of $\gamma_1 \cdot \gamma_2$ and $\gamma_1 \cdot \hat{P} \gamma_2 \cdot \hat{P}$. Scalar interactions \tilde{S}_i depend on scalar invariant $S(r)$ and also vector invariant $A(r)$. However, $\tilde{S}_i(S(r) = 0, A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2) = 0$. The Pauli reduction of TBDE leads to a covariant Schrödinger-like equation (SLE) for the relative motion with explicit spin-dependent potential Φ . In the c.m. system:

$$\begin{aligned} &\{\mathbf{p}^2 + \Phi(\mathbf{r}, m_1, m_2, w, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)\}\psi_+ \\ &= \{\mathbf{p}^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + 2\varepsilon_w V - V^2 + \Phi_D \\ &\quad + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \Phi_{SO} + \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} \mathbf{L} \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \Phi_{SOT} \\ &\quad + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \Phi_{SS} + (3\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \Phi_T \\ &\quad + \mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \Phi_{SOD} + i\mathbf{L} \cdot \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 \Phi_{SOX}\}\psi_+ \\ &= b^2\psi_+, \end{aligned} \tag{4}$$

where ψ_+ is a 4-component spinor subcomponent of 16 component spinor ψ . Note that the SLE shares the spin-independent parts discussed earlier. The TBDE and the equivalent SLE possess important and desirable features:

1. TBDE reduce to one-body Dirac form for m_1 when $m_2 \rightarrow \infty$ (the Salpeter equation does not have this property).
2. SLE goes into the nonrelativistic Schrödinger equation in limit of weak binding and small speeds.
3. SLE can be solved nonperturbatively for QED bound states of positronium and muonium as well as QCD meson bound states since: a) every term in Φ is less attractive than $-(1/4)r^2$ (also no $\delta(\mathbf{r})$ or attractive $1/r^3$ potentials) b) the covariant Dirac formalism introduces natural cutoff factors that smooth out singular spin-dependent interactions, no need to introduce them by hand as in other approaches.

4. The \tilde{A}_i^μ , \tilde{S}_i in the TBDE are directly related to perturbative QFT and for mesons may be introduced semiphenomenologically through $A(r)$ and $S(r)$ and $V(r)$.
5. SLE have been tested analytically and numerically against the known QED perturbative spectrum. The (nonperturbative) successes for the QED spectrum gives confidence that a numerical treatment of the SLE in QCD accurately reflects the physical implications of chosen invariant A, V, S .
6. TBDE provide covariant 3D framework in which the local potential approximation consistently fulfills the requirements of gauge invariance in QED[17].
7. SLE with $\Phi(A = -\alpha/r, V = 0, S = 0)$ is responsible for accurate QED spectral results.

For QCD spectra we use $\Phi(A(r) \neq -\alpha/r, V(r) \neq 0, S(r) \neq 0)$ with A, V, S obtained from the static Adler-Piran potential.

3.1 Two Body Dirac Equations for Meson Spectroscopy

Adler and Piran [18] developed a potential for heavy static quarks from QCD. Their model resembles nonlinear electrostatics with a nonlinear effective dielectric constant. Integrating their solution fixes all parameters in their model apart from a mass scale Λ and an “integration constant” U_0 ,

$$\begin{aligned} (9/16\pi) \nabla \cdot [\ln(\mathbf{E}^2/\Lambda^2)] \mathbf{E} &= 4\pi Q [\delta(\mathbf{x} - \mathbf{x}_1) - \delta(\mathbf{x} - \mathbf{x}_2)], \\ \int \mathbf{E}^2(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2) d^3x &= V_{AP}(|\mathbf{x}_1 - \mathbf{x}_2|) = \Lambda(U(\Lambda |\mathbf{x}_1 - \mathbf{x}_2|) + U_0). \end{aligned}$$

We divide V_{AP} invariants A, V and S that appear in SLE so that

$$V_{AP}(r) + V_{coul} = \Lambda(U(\Lambda r) + U_0) + \frac{e_1 e_2}{r} = A + V + S.$$

The V_{AP} incorporates asymptotic freedom through

$$\Lambda U(\Lambda r \ll 1) \sim 1/(r \ln \Lambda r),$$

and confinement through linear and subdominant potential terms,

$$\Lambda U(\Lambda r \gg 1) \sim \Lambda^2 r, \quad \Lambda \ln \Lambda r, \quad \sqrt{\frac{\Lambda}{r}}, \quad \frac{k}{r}, \quad a\Lambda.$$

We compute the best fit to the meson spectrum using this division of Adler-Piran potential[3]:

$$\begin{aligned} A &= \exp(-\beta \Lambda r) [V_{AP} - \frac{k}{r}] + \frac{k}{r} + \frac{e_1 e_2}{r}, \\ V + S &= V_{AP} + \frac{e_1 e_2}{r} - A = (V_{AP} - \frac{k}{r})(1 - \exp(-\beta \Lambda r)) \equiv \mathcal{U}. \end{aligned}$$

Thus, V_{AP} is covariantly incorporated into the SLE by treating the short distance portion as purely electromagnetic-like ($\sim A \gamma_{1\mu} \gamma_2^\mu$). The attractive ($k < 0$) QCD-Coulomb-like part of V_{AP} ($\Lambda r \gg 1$) is assigned completely to electromagnetic-like part A . The exponential factor $\exp(-\beta \Lambda r)$ gradually turns off A at long distances except for $k/r + e_1 e_2/r$. The scalar and timelike portions (S and

V) gradually turn on, becoming fully responsible for the linear confining and subdominant terms at long distance. The three invariants A, V, S depend on three parameters: Λ, U_0 , and β . We introduce a fourth parameter ξ which divides the confining portion \mathcal{U} into scalar and time-like vector parts:

$$\begin{aligned} e_2 S &= \xi \mathcal{U} = \xi (V_{AP} - k/r)(1 - \exp(-\beta \Lambda r)), \\ V &= \mathcal{U} - S = (1 - \xi)(V_{AP} - k/r)(1 - \exp(-\beta \Lambda r)). \end{aligned}$$

The best fit parameter values are[3]

Parameter	Best fit values
m_b	4.953 GeV
m_c	1.585 GeV
m_s	0.3079 GeV
m_u	0.0985 GeV
m_d	0.1045 GeV
Λ	0.2255 GeV
ΛU_0	1.770 GeV
$\beta \Lambda$	0.994 GeV=1/(0.198 fermi)
ξ	0.704

and indicate that the confining portion begins to dominate at about 0.2 femis and that the scalar interaction makes up about 70% of the confining part of the potential.

The experimental and theoretical values of the meson masses are given in GeV with the errors given in MeV in parentheses. For $u\bar{d}$ mesons the above parameters yield

$u\bar{d}$ mesons	Exp.	Th.	χ^2 .
$\pi : u\bar{d} \ 1^1 S_0$	0.140(0.0)	0.134	0.3
$\rho : u\bar{d} \ 1^3 S_1$	0.775(0.4)	0.781	0.2
$b_1 : u\bar{d} \ 1^1 P_1$	1.230(3.2)	1.243	0.2
$a_1 : u\bar{d} \ 1^3 P_1$	1.230(40.)	1.320	0.1
$\pi : u\bar{d} \ 2^1 S_0$	1.300(100)	1.435	0.0
$a_2 : u\bar{d} \ 1^3 P_2$	1.318(0.6)	1.310	0.5
$\rho : u\bar{d} \ 2^3 S_1$	1.465(25.)	1.684	0.8
$a_0 : u\bar{d} \ 1^3 P_0$	1.474(19.)	1.024	5.6
$b_2 : u\bar{d} \ 1^1 D_2$	1.672(3.2)	1.763	7.2

Ground state fits are good but some of the radial and some orbital excitation are off. Note that if we replace $| a_0 : u\bar{d} \ 1^3 P_0 \ 1.474(19.) \ 1.024 \rangle$ by $| a_0 : u\bar{d} \ 1^3 P_0 \ 0.980(20.) \ 1.024 \rangle$ the fit is much better. In this case we treat the 1.474 GeV meson as a first radial excitation. This leads to $| a_0 : u\bar{d} \ 2^3 P_0 \ 1.474(19.) \ 1.784 \rangle$. The fit is better on both accounts.

Fits to $s\bar{u}, s\bar{d}$ mesons are good for the ground states with some exceptions on the radial and orbital excitaions. The listed 1.425 meson would probably be better fit as a radial excitation.

$s\bar{u}, s\bar{d}$ Mesons	Exp.	Th.	χ^2 -Th.
$K^- : s\bar{u} \ 1^1S_0$	0.494(0.0)	0.519	6.4
$K^0 : s\bar{d} \ 1^1S_0$	0.498(0.0)	0.520	5.0
$K^{*-} : s\bar{u} \ 1^3S_1$	0.892(0.3)	0.896	0.2
$K^{*0} : s\bar{d} \ 1^3S_1$	0.896(0.3)	0.897	0.0
$K^- : s\bar{u} \ 1^1P_1$	1.272(7.0)	1.339	0.9
$K^{*-} : s\bar{u} \ 1^3P_1$	1.403(7.0)	1.359	0.4
$K^{*-} : s\bar{u} \ 2^3S_1$	1.414(15.)	1.706	3.8
$K^{*-} : s\bar{u} \ 1^3P_0$	1.425(50.)	1.079	0.5
$K^{*-} : s\bar{u} \ 1^3P_2$	1.426(1.5)	1.404	1.4
$K^{*0} : s\bar{d} \ 1^3P_2$	1.432(1.3)	1.405	2.8
$K^- : s\bar{u} \ 2^1S_0$	1.460(40.)	1.476	0.0
$K^{*-} : s\bar{u} \ 1^3D_1$	1.717(27.)	1.837	0.2
$K^- : s\bar{u} \ 1^1D_2$	1.773(8.0)	1.803	0.1
$K^{*-} : s\bar{u} \ 1^3D_3$	1.776(7.0)	1.792	0.0
$K^{*-} : s\bar{u} \ 1^3D_2$	1.816(13.)	1.795	0.0

The $s\bar{s}$ family of mesons shows a good fit to the ground state and usual mix of results to the spin-orbit triplet.

$s\bar{s}$ Mesons	Exp.	Th.	χ^2 -Th.
$\phi : s\bar{s} \ 1^3S_1$	1.019(0.0)	1.013	0.4
$\phi : s\bar{s} \ 1^3P_0$	1.370(100)	1.175	0.0
$\phi : s\bar{s} \ 1^3P_1$	1.518(5.0)	1.437	2.5
$\phi : s\bar{s} \ 1^3P_2$	1.525(5.0)	1.506	0.1
$\phi : s\bar{s} \ 2^3S_1$	1.680(20.)	1.875	0.9
$\phi : s\bar{s} \ 1^3D_3$	1.854(7.0)	1.879	0.1
$\phi : s\bar{s} \ 2^3P_2$	2.011(70)	2.128	0.0

The $c\bar{u}, c\bar{d},$ and $c\bar{s}$ mesons display good fits for ground states

$c\bar{u}, c\bar{d}, c\bar{s}$ Mesons	Exp.	Th.	χ^2 -Th.
$D^0 : c\bar{u} \ 1^1S_0$	1.865(0.2)	1.876	1.1
$D^+ : c\bar{d} \ 1^1S_0$	1.870(0.2)	1.883	1.7
$D^{*0} : c\bar{u} \ 1^3S_1$	2.007(0.2)	2.007	0.0
$D^{*+} : c\bar{d} \ 1^3S_1$	2.010(0.2)	2.013	0.1
$D^{*0} : c\bar{u} \ 1^3P_0$	2.352(50.)	2.221	0.1
$D^{*+} : c\bar{d} \ 1^3P_0$	2.403(14.)	2.230	1.5
$D^+ : c\bar{d} \ 1^3P_2$	2.460(3.0)	2.414	2.1
$D^{*0} : c\bar{u} \ 1^3P_2$	2.461(1.6)	2.409	7.7

Note that the $c\bar{s}$ spin orbit triplet gives reasonable fits as opposed to $s\bar{s}$

$D_s : c\bar{s} \ 1^1S_0$	1.968(0.3)	1.974	0.3
$D_s^* : c\bar{s} \ 1^3S_1$	2.112(0.5)	2.119	0.4
$D_s^* : c\bar{s} \ 1^3P_0$	2.318(0.6)	2.340	3.5
$D_s : c\bar{s} \ 1^1P_1$	2.535(0.3)	2.499	11.6
$D_s^* : c\bar{s} \ 1^3P_2$	2.573(0.9)	2.532	8.9
$D_s^* : c\bar{s} \ 2^3S_1$	2.690(7.0)	2.702	0.0

The charmonium family is given by

$c\bar{c}$ Mesons	Exp.	Th.	χ^2 -Th.
$\eta_c : c\bar{c} \ 1^1S_0$	2.980(1.2)	2.973	0.2
$J/\psi(1S) : c\bar{c} \ 1^3S_1$	3.097(0.0)	3.128	9.7
$\chi_0 : c\bar{c} \ 1^3P_0$	3.415(0.3)	3.397	3.0
$\chi_1 : c\bar{c} \ 1^3P_1$	3.511(0.1)	3.505	0.4
$h_1 : c\bar{c} \ 1^1P_1$	3.526(0.3)	3.523	0.1
$\chi_2 : c\bar{c} \ 1^3P_2$	3.556(0.1)	3.557	0.0
$\eta_c : c\bar{c} \ 2^1S_0$	3.637(4.0)	3.602	0.7
$\psi(2S) : c\bar{c} \ 2^3S_1$	3.686(0.0)	3.689	0.1
$\psi(1D) : c\bar{c} \ 1^3D_1$	3.773(0.4)	3.807	0.9
$\chi_2 : c\bar{c} \ 2^3P_2$	3.929(5.0)	3.983	1.1
$\psi(3S) : c\bar{c} \ 3^3S_1$	4.039(10.)	4.092	0.3
$\psi(2D) : c\bar{c} \ 2^3D_1$	4.153(3.0)	4.169	0.3
$\psi(4S) : c\bar{c} \ 4^3S_1$	4.421(4.0)	4.426	0.0
$\psi(3D) : c\bar{c} \ 3^3D_1$	4.421(4.0)	4.483	2.3

The overall fit is good with the worst fit meson of the family is the J/ψ . The $b\bar{u}$, $b\bar{d}$, $b\bar{s}$, $b\bar{c}$ mesons

$b\bar{u}, b\bar{d} \ b\bar{s}$ Mesons	Exp.	Th.	χ^2 -Th.
$B^- : b\bar{u} \ 1^1S_0$	5.279(0.3)	5.283	0.2
$B^0 : b\bar{d} \ 1^1S_0$	5.280(0.3)	5.284	0.2
$B^{*-} : b\bar{u} \ 1^3S_1$	5.325(0.5)	5.333	0.5
$B^{*-} : b\bar{u} \ 1^3P_2$	5.747(2.9)	5.687	3.8
$B_s^0 : b\bar{s} \ 1^1S_0$	5.366(0.6)	5.367	0.0
$B_s^{*0} : b\bar{s} \ 1^3S_1$	5.413(1.3)	5.430	1.0
$B_s^{*0} : b\bar{s} \ 1^3P_1$	5.829(0.7)	5.792	9.4
$B_s^{*0} : b\bar{s} \ 1^3P_2$	5.840(0.6)	5.805	9.0
$B_c^- : b\bar{c} \ 1^1S_0$	6.276(21.)	6.251	0.4

display very good results for the ground states. Finally for the $b\bar{b}$ mesons, even though the overall fit is a good (one exception is the 3rd radial excitation), the spin – spin splitting of the ground state is, oddly, not as good as for the lighter mesons.

$b\bar{b}$ Mesons	Exp.	Th.	χ^2 -Th.
$\eta_b : b\bar{b} \ 1^1S_0$	9.389(4.0)	9.330	2.0
$\Upsilon(1S) : b\bar{b} \ 1^3S_1$	9.460(0.3)	9.444	2.6
$\chi_{b0} : b\bar{b} \ 1^3P_0$	9.859(0.4)	9.834	5.6
$\chi_{b1} : b\bar{b} \ 1^3P_1$	9.893(0.3)	9.886	0.4
$\chi_{b2} : b\bar{b} \ 1^3P_2$	9.912(0.3)	9.920	0.6
$\Upsilon(2S) : b\bar{b} \ 2^3S_1$	10.023(0.3)	10.022	0.0
$\Upsilon(D) : b\bar{b} \ 2^3D_2$	10.161(0.6)	10.179	2.3
$\chi_{b0} : b\bar{b} \ 2^3P_0$	10.232(0.4)	10.229	0.1
$\chi_{b1} : b\bar{b} \ 2^3P_1$	10.255(0.5)	10.262	0.4
$\chi_{b2} : b\bar{b} \ 2^3P_2$	10.269(0.4)	10.286	2.5
$\Upsilon(3S) : b\bar{b} \ 3^3S_1$	10.355(0.6)	10.368	1.2
$\Upsilon(4S) : b\bar{b} \ 4^3S_1$	10.579(1.2)	10.633	11.7
$\Upsilon(5S) : b\bar{b} \ 5^3S_1$	10.865(8.0)	10.857	0.0
$\Upsilon(6S) : b\bar{b} \ 6^3S_1$	11.019(8.0)	11.055	0.2

3.2 Application of Two Body Dirac Equations to Baryon Spectroscopy

Sazdjian [19] combined three pairs of interacting quarks into a single relativistically covariant three body equation for bound states, having a Schrödinger-like structure. There is no space to develop his approach here. We say a few words about the analogy of his results to that of the two body equations. Recall that for two bodies we have the results

$$\begin{aligned}
\mathcal{H}_1\psi &= [p_1^2 + m_1^2 + \Phi_{12}] \psi = 0, \\
\mathcal{H}_2\psi &= [p_2^2 + m_2^2 + \Phi_{12}] \psi = 0, \\
\varepsilon_1 &= [w + (m_1^2 - m_2^2) / (\varepsilon_1 + \varepsilon_2)] / 2, \\
\varepsilon_2 &= [w + (m_2^2 - m_1^2) / (\varepsilon_1 + \varepsilon_2)] / 2, \\
\varepsilon_1 + \varepsilon_2 &= w, \\
[\mathcal{H}_1, \mathcal{H}_2] \psi &= 0 \rightarrow \Phi_{12} = \Phi_{12}(x_{12\perp}), \\
(p_{1\perp}^2 + \Phi_{12}) \psi &= (\varepsilon_1^2 - m_1^2) \psi = (\varepsilon_2^2 - m_2^2) \psi = b^2(w) \psi.
\end{aligned}$$

For three bodies, speaking heuristically

$$\begin{aligned}
\mathcal{H}_1\psi &= [p_1^2 + m_1^2 + \Phi_{12} + \Phi_{31}] \psi = 0, \\
\mathcal{H}_2\psi &= [p_2^2 + m_2^2 + \Phi_{23} + \Phi_{12}] \psi = 0, \\
\mathcal{H}_3\psi &= [p_3^2 + m_3^2 + \Phi_{31} + \Phi_{23}] \psi = 0, \\
\varepsilon_1 &= [w + (m_1^2 - m_2^2) / (\varepsilon_1 + \varepsilon_2) + (m_1^2 - m_3^2) / (\varepsilon_1 + \varepsilon_3)] / 3, \\
\varepsilon_2 &= [w + (m_2^2 - m_3^2) / (\varepsilon_2 + \varepsilon_3) + (m_2^2 - m_1^2) / (\varepsilon_2 + \varepsilon_1)] / 3, \\
\varepsilon_3 &= [w + (m_3^2 - m_1^2) / (\varepsilon_3 + \varepsilon_1) + (m_3^2 - m_2^2) / (\varepsilon_3 + \varepsilon_2)] / 3, \\
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 &= w \\
\Phi_{12} &= \Phi_{12}(x_{12\perp}), \Phi_{23} = \Phi_{23}(x_{23\perp}), \Phi_{31} = \Phi_{31}(x_{31\perp}), \\
x_{ij\perp}^\mu &= (x_i^\mu - x_j^\mu) + \hat{P}^\mu \hat{P} \cdot (x_i - x_j),
\end{aligned}$$

where $P = \sum_{i=1}^N p_i$ is the *total* momentum (not $p_i + p_j$). Unlike the case of two bodies, the $x_{ij\perp}^\mu$ dependence is obtained by a more roundabout approach.

The sum three body Schrödinger-like which we adopt from his approach is[5]

$$\begin{aligned}
\mathcal{H}\psi &\equiv \frac{1}{F} \left(\frac{p_{1\perp}^2 + \Phi_{12} + \Phi_{13}}{2\varepsilon_1(w, m_1, m_2, m_3)} + \frac{p_{2\perp}^2 + \Phi_{23} + \Phi_{12}}{2\varepsilon_2(w, m_1, m_2, m_3)} + \frac{p_{3\perp}^2 + \Phi_{31} + \Phi_{23}}{2\varepsilon_3(w, m_1, m_2, m_3)} \right) \psi \\
&= (w - m_1 - m_2) \psi,
\end{aligned} \tag{5}$$

in which $F = F(w, m_1, m_2, m_3)$ is a complicated function of the invariant w and the three masses. We choose Φ_{ab} to have the same functional dependence on S and A as in SLE form of TBDE

$$\begin{aligned}
&\Phi_{ab}(\mathbf{r}_{ab}, m_a, m_b, w_{ab}, \boldsymbol{\sigma}_a, \boldsymbol{\sigma}_b) \\
&= 2m_{w_{ab}} S + S^2 + 2\varepsilon_{w_{ab}} A - A^2 + 2\varepsilon_{w_{ab}} V - V^2 + \Phi_D \\
&\quad + \mathbf{L}_{ab} \cdot (\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_b) \Phi_{SO} + \boldsymbol{\sigma}_a \cdot \hat{\mathbf{r}}_{ab} \boldsymbol{\sigma}_b \cdot \hat{\mathbf{r}}_{ab} \mathbf{L}_{ab} \cdot (\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_b) \Phi_{SOT} \\
&\quad + \boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b \Phi_{SS} + (3\boldsymbol{\sigma}_a \cdot \hat{\mathbf{r}}_{ab} \boldsymbol{\sigma}_b \cdot \hat{\mathbf{r}}_{ab} - \boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b) \Phi_T \\
&\quad + \mathbf{L}_{ab} \cdot (\boldsymbol{\sigma}_a - \boldsymbol{\sigma}_b) \Phi_{SOD} + i \mathbf{L}_{ab} \cdot \boldsymbol{\sigma}_a \times \boldsymbol{\sigma}_b \Phi_{SOX}, \\
&w_{ab} = \varepsilon_a + \varepsilon_b.
\end{aligned}$$

Note in the NR limit, $F \rightarrow 1$ and $\varepsilon_i \rightarrow m_i$. This spin-dependent potential used in the three-body bound state equation is not a result of the reduction of some set of three-body Dirac equations. Rather, it is the two-body SLE quasipotential inserted by hand as an addition into free Klein-Gordon forms. The equation is solved by variational approach[5].

For the ground state octet the spectral results below (one set of parameters for the entire baryon spectrum) indicate a good fit for the nucleons, high for the strangeness caring baryons but low for the Λ .

Baryon	J	L	S	Th. Mass (MeV)	Exp. Mass(MeV)	Exp-Th.(MeV)
p	1/2	0	1/2	947	938	-9
n	1/2	0	1/2	948	939	-9
Σ^+	1/2	0	1/2	1250	1189	-61
Σ^0	1/2	0	1/2	1261	1192	-68
Σ^-	1/2	0	1/2	1271	1197	-73
Ξ^0	1/2	0	1/2	1373	1314	-58
Ξ^-	1/2	0	1/2	1378	1321	-57
Λ^0	1/2	0	1/2	1082	1125	43

For the ground state decimet the higher strangeness particles lie lower instead of higher as with the octet

Baryon	J	L	S	Th. Mass (MeV)	Exp. Mass(MeV)	Exp-Th.(MeV)
Δ^{++}	3/2	0	3/2	1249	1232	-17
Δ^+	3/2	0	3/2	1250	1232	-18
Δ^0	3/2	0	3/2	1251	1232	-19
Δ^-	3/2	0	3/2	1252	1232	-20
$\Sigma^+(1390)$	3/2	0	3/2	1384	1383	-1
$\Sigma^0(1390)$	3/2	0	3/2	1385	1384	-1
$\Sigma^-(1390)$	3/2	0	3/2	1387	1387	0
$\Xi^0(1530)$	3/2	0	3/2	1501	1531	30
$\Xi^-(1530)$	3/2	0	3/2	1507	1535	28
Ω^-	3/2	0	3/2	1609	1672	63

For the orbital and radial excitations the results are mixed but special note is taken for the good fit to the $\Lambda(1405)$

Baryon	J	L	S	Th. Mass (MeV)	Exp. Mass(MeV)	Exp-Th.
$N(1440)$	1/2	0	1/2	1557	1420-1470	-117
$\Lambda(1600)$	1/2	0	1/2	1677	1560-1700	-77
$\Sigma(1660)$	1/2	0	1/2	1672	1630-1690	12
$\Sigma(1880)$	1/2	0	1/2	1709	1800-1960	171
$\Xi(1690)$	1/2	0	1/2	1784	1680-1700	-94
$\Delta(1600)$	3/2	0	3/2	1521	1550-1700	78
$N(1535)$	1/2	1	1/2	1549	1525-1545	-14
$\Delta(1620)$	1/2	1	1/2	1542	1600-1660	78
$\Lambda(1405)$	1/2	1	1/2	1410	1402-1410	-4
$\Lambda(1670)$	1/2	1	1/2	1671	1660-1680	-1
Baryon	J	L	S	Th. Mass (MeV)	Exp. Mass(MeV)	Exp-Th.
$N(1650)$	1/2	1	3/2	1566	1645-1670	84
$\Sigma(1750)$	1/2	1	3/2	1644	1730-1800	121
$\Lambda(1800)$	1/2	1	3/2	1658	1720-1850	142
$N(1520)$	3/2	1	1/2	1551	1515-1525	-31
$\Delta(1700)$	3/2	1	1/2	1546	1670-1750	154
$\Sigma(1670)$	3/2	1	1/2	1679	1665-1685	-4
$\Lambda(1520)$	3/2	1	1/2	1680	1518-1521	-160
$\Lambda(1690)$	3/2	1	1/2	1670	1685-1695	20
$\Xi(1820)$	3/2	1	1/2	1777	1818-1828	43
$N(1700)$	3/2	1	3/2	1568	1650-1750	132
$\Sigma(1775)$	5/2	1	3/2	1661	1770-1780	114
$N(1675)$	5/2	1	3/2	1615	1670-1680	59
$\Lambda(1830)$	5/2	1	3/2	1641	1810-1830	189
$\Xi(1950)$	5/2	1	3/2	1757	1935-1965	192

Finally we have the baryons that involve the charmed and bottom quarks with mixed results.

Baryon	J	L	S	Th. Mass (MeV)	Exp. Mass(MeV)	Exp-Th.(MeV)
$\Sigma_c^{++}(2455)$	1/2	0	1/2	2385	2454	68
$\Sigma_c^{++}(2520)$	3/2	0	3/2	2551	2520	-31
$\Lambda_c^+(2286)$	1/2	0	1/2	2382	2286	-96
$\Lambda_c^+(2595)$	1/2	1	1/2	2415	2595	180
$\Xi_c^+(2467)$	1/2	0	1/2	2561	2467	-94
$\Xi_c^0(2470)$	1/2	0	1/2	2562	2470	-92
$\Xi_c^+(2645)$	3/2	0	3/2	2598	2645	46
$\Xi_c^+(2790)$	1/2	1	3/2	2661	2790	129
$\Xi_c^+(2815)$	3/2	1	3/2	2707	2815	108
$\Omega_c^0(2695)$	1/2	0	1/2	2732	2695	-37
$\Omega_c^0(2770)$	3/2	0	3/2	2745	2770	25
$\Sigma_b^+(5829)$	3/2	0	3/2	5800	5829	29
$\Sigma_b^-(5836)$	3/2	0	3/2	5851	5836	-15
$\Xi_b^0(5790)$	1/2	0	1/2	5854	5790	-64
$\Omega_b^-(6071)$	1/2	0	1/2	6032	6071	39

3.3 Two Body Dirac Equations for QED

The SLE given in Eq. (4) can be used for QED as well as QCD bound states. For meson spectroscopy, the three invariant functions $S(r)$, $A(r)$, and $V(r)$ fix $\Phi(\mathbf{r}, m_1, m_2, w, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$. For QED bound states $S(r) = V(r) = 0$ and

$$A(r) = -\frac{\alpha}{r} \equiv -\frac{\alpha}{\sqrt{x_\perp^2}} = -\frac{\alpha}{|\mathbf{r}|} (\text{in c.m.}).$$

The QED spectral results follow from solving numerically or analytically, the radial forms of SLE. For equal mass spin singlet, the attractive spin-spin quasipotential $(-3\Phi_{SS})$ exactly cancels repulsive Darwin quasipotential Φ_D , giving an eigenvalue equation for 1J_J states

$$\left\{ -\frac{d^2}{dr^2} + \frac{J(J+1)}{r^2} + 2\varepsilon_w A - A^2 \right\} u_0 = b^2 u_0.$$

For point electron and positron $A = -\alpha/r \rightarrow$

$$\left\{ -\frac{d^2}{dr^2} + \frac{J(J+1)}{r^2} - \frac{2\varepsilon_w \alpha}{r} - \frac{\alpha^2}{r^2} \right\} u_0 = b^2 u_0.$$

This has ground state analytic spectral solution [20] with accepted $O(\alpha^4)$ perturbative expansion

$$w = m \sqrt{2 + 2/\sqrt{1 + \frac{\alpha^2}{\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha^2}\right)^2}}} = 2m - \frac{m\alpha^2}{4} - \frac{21m\alpha^4}{64} + O(\alpha^6)...$$

At short distance, the SLE equation takes on the limiting form

$$\left\{ -\frac{d^2}{dr^2} + \frac{J(J+1)}{r^2} - \frac{\alpha^2}{r^2} \right\} u = 0.$$

Since $J(J+1) - \alpha^2 > -1/4$ the effective potential is nonsingular, implying a well defined solutions. Numerical solutions of the eigenvalue equations yield spectra agreeing with standard perturbative $O(\alpha^4)$ results. E.g. for the singlet ground state of positronium[6]

$$\begin{aligned} \text{numerical binding energy} &= -6.8033256279 \text{ eV}, \\ \text{vs } m(-\alpha^2/4 - 21\alpha^4/64) &= -6.8033256719 \text{ eV}. \end{aligned}$$

The difference is on order of $m\alpha^6$. For the triplet ground state of positronium

$$\begin{aligned} \text{numerical binding energy} &= -6.8028426132 \text{ eV}, \\ \text{vs } m(-\alpha^2/4 + \alpha^4/192) &= -6.8028426636 \text{ eV}. \end{aligned}$$

The difference is also on order of $m\alpha^6$. This does not include annihilation diagram (nor radiative corrections).

These two results are from a very extensive list of numerically computed spectral [6] showing TBDE passes crucial tests, ones not demonstrated in any other relativistic bound state formalism. Sommerer, Spence and Vary [21] have found a particular quasipotential formalism that does give such agreement, but only for the ground state. They also demonstrate that several prominent two-body relativistic bound state formalisms (including the Blankenbecler-Sugar formalism [22], and the formalism of Gross [23]) fail this important test. The importance of numerical tests of the formalisms is not for QED, but rather as a reliability test for use of the corresponding formalisms, e.g. Coulomb gauge BSE in QCD [24]. If failure occurs in their applications to QED bound states this brings into question the spectral results of similar nonperturbative (i.e. numerical) approaches applied to QCD bound states.

3.3.1 Peculiar Singlet Positronium Bound States

These last two topics are on peculiar solutions of the TBDE and are speculative with new phenomena predicted for the positronium system. We begin by a critical examination of the bound state equation for point e^+ and e^- .

$$\left\{ -\frac{d^2}{dr^2} - \frac{2\varepsilon_w \alpha}{r} - \frac{\alpha^2}{r^2} \right\} u_0 = b^2 u_0 \quad (6)$$

Based on this equation [7] we find: a new positronium bound state with a large (300 KeV) binding energy derived from an exact solution of the above equation. The new positronium bound state would result from a metastable two-photon decay of the usual positronium ground state which has a binding energy of about 6.8 eV. It then annihilates promptly into 2 photon with c.m. energy of 700 KeV. The existence of this new positronium state would thus be a distinctive 4 photon decay signature of the usual singlet positronium ground state. The size of the new positronium bound state is on the order of an electron's Compton wave length.

Eq.(6) has the short distance ($r \ll \alpha/2\varepsilon_w$) behavior

$$\left\{ -\frac{d^2}{dr^2} - \frac{\alpha^2}{r^2} \right\} u = 0,$$

with solutions called usual and peculiar,

$$\begin{aligned} u_+ &\sim r^{\lambda_++1}; \lambda_+ = (-1 + \sqrt{1 - 4\alpha^2})/2; + \text{ usual} \\ u_- &\sim r^{\lambda_-+1}; \lambda_- = (-1 - \sqrt{1 - 4\alpha^2})/2; - \text{ peculiar.} \end{aligned}$$

With these behaviors, the probability is finite for both signs

$$\psi_\pm^2 d^3r = \frac{u_\pm^2}{r^2} r^2 dr d\Omega = u_\pm^2 dr d\Omega = r^{(1 \pm \sqrt{1 - 4\alpha^2})} dr d\Omega.$$

Both of these behaviors are quantum mechanically acceptable near the origin. If $L \neq 0$ so that $L(L+1) - \alpha^2 > 0$ or the electron is not a point particle then the peculiar solution not physically admissible.

Both 1S_0 bound state solutions can be obtained analytically. The respective sets of eigenvalues for total invariant c.m. energy (mass) $w_{\pm n}$ (n is principle quantum #)

$$w_{\pm n} = m \sqrt{2 + 2/\sqrt{1 + \alpha^2/(n \pm \sqrt{1/4 - \alpha^2} - 1/2)^2}}.$$

The usual state ground eigenvalue gives standard QED perturbative results thru order α^4

$$w_{+n} = 2m - m\alpha^2/4 - 21m\alpha^4/64 + O(\alpha^6), \quad n = 1, 2, 3, \dots$$

The peculiar ground state $n = 1$ has mass

$$w_{-1} = m \sqrt{2 + 2/\sqrt{1 + \alpha^2/(1/2 - \sqrt{1/4 - \alpha^2})^2}} \sim \sqrt{2}m\sqrt{1 + \alpha},$$

which represents very tight binding energy on order 300 KeV for an e^+e^- state. Its weak coupling limit is antiintuitive, having a total c.m. energy $\rightarrow \sqrt{2}m$ instead of $2m$.

The two $n = 1$ wave functions have the respective forms

$$\begin{aligned} u_+(r) &= c_+ r^{\lambda_++1} \exp(-\kappa_+ \varepsilon_{w_+} \alpha r), \quad \kappa_+ = \frac{2}{1 + \sqrt{1 - 4\alpha^2}} = \frac{1}{\lambda_+ + 1}, \\ u_-(r) &= c_- r^{\lambda_-+1} \exp(-\kappa_- \varepsilon_{w_-} \alpha r), \quad \kappa_- = \frac{2}{1 - \sqrt{1 - 4\alpha^2}} = \frac{1}{\lambda_- + 1}. \end{aligned}$$

Since they are both zero node solutions, they are not orthogonal (although the inner product is small, $\sim 1/1000$)

$$\langle u_- | u_+ \rangle = \int_0^\infty dr u_+(r) u_-(r) \sim \alpha^{3/2} \neq 0.$$

How do we reconcile this with the expected orthogonality of the eigenfunctions of a self-adjoint operator corresponding to different eigenvalues? One can show that the second derivative is not self-adjoint in this context! However, we emphasize the fact that both the set of usual and peculiar states are quantum mechanically admissible states. We admit both types of physical states into a larger Hilbert space by introducing a new observable $\hat{\zeta}$ with a quantum number which we call "peculiarity" allowing the mass operator to be self-adjoint, and the set of physically allowed states become a complete set. In particular such that

$$\begin{aligned}\hat{\zeta}\chi_+ &= \zeta\chi_+ \text{ with eigenvalue } \zeta = +1, \text{ usual positronium,} \\ \hat{\zeta}\chi_- &= \zeta\chi_- \text{ with eigenvalue } \zeta = -1, \text{ peculiar positronium,}\end{aligned}$$

with the corresponding spinor wave function χ_ζ assigned to the states so that a usual state is represented by the peculiarity spinor χ_+ ,

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and a peculiar state is represented by the peculiarity spinor χ_-

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

With this introduction, a general wave function can be expanded in terms of the complete set of basis functions $\{u_{+n}, u_{-n}\}$ as

$$\Psi = \sum_{\zeta n} a_{\zeta n} u_{\zeta n} \chi_\zeta,$$

where n represents spin and spatial quantum numbers and ζ the peculiarity. The variational principle applied to

$$\langle H \rangle = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle},$$

would lead to

$$\begin{aligned}Hu_{+n}\chi_+ &= -\kappa_{+n}^2 u_{+n}\chi_+, \\ Hu_{-n}\chi_- &= -\kappa_{-n}^2 u_{-n}\chi_-.\end{aligned}$$

Thus the introduction of the peculiarity quantum number resolves the problem of the over-completeness property of the basis states and the non-self-adjoint property of the mass operator.

If the peculiarity quantum number is strictly conserved it would be impossible for the usual positronium ground state ($1S_u$) to decay to the peculiar ground state ($1S_p$) and usual ground state would only undergo the usual two photon annihilation in about 10^{-10} sec. We consider possible evidence that this quantum number is not conserved for the full Hamiltonian. In that case we could have that the usual ground state undergo a metastable decay into the peculiar ground state by emitting two photons. We obtain a lifetime of

$$\tau_{1S_u \rightarrow 1S_p + 2\gamma} \sim \frac{\tau_{1S_u \rightarrow 2\gamma} \pi^4}{2.55\alpha^2} = 9.0 \times 10^{-5} \text{sec.} \quad (7)$$

and a two photon annihilation lifetime $1S_p$ on the order of

$$\tau_{1S_p \rightarrow 2\gamma} \sim \frac{\tau_{1S_u \rightarrow 2\gamma}}{\alpha^3} \sim 10^{-16} \text{sec.}$$

This implies that we would see 4γ as the signature of the production and decay of the peculiar positronium ground state. We obtain a small branching ratio compared with the annihilation of the usual positronium ground state into two 500 KeV photons. Failure to find the peculiar state at the predicted energy would imply that electron and positron are not point-like or that radiative corrections lead to less attractive potentials that do not give quantum mechanically acceptable double roots of the leading short distance behavior.

3.3.2 New Peculiar 3P_0 e^+e^- QED Resonances

A closely related state to peculiar positronium is a pure QED e^+e^- resonance from the highly attractive magnetic spin-orbit interaction between point electron and positron in the 3P_0 angular momentum state. We find in particular a resonance at about 28 MeV with a narrow width of about 15 KeV[7].

The angular momentum barrier is overwhelmed by relativistic effective potentials at very short distances. The SLE for the 3P_0 state is

$$\begin{aligned} \left\{ -\frac{d^2}{dr^2} + \frac{2}{r^2} + \Phi(r) \right\} u &= b^2 u, \\ \frac{2}{r^2} + \Phi(r) &= \frac{2}{(r + 2\alpha/w)^2} - \frac{2\varepsilon_w\alpha}{r} - \frac{\alpha^2}{r^2}. \end{aligned} \quad (8)$$

In first term on the right hand side, we see that the angular momentum barrier $2/r^2$ is overwhelmed by the net effects of the magnetic interactions (spin-orbit, spin-spin, tensor and Darwin interactions) and the $-\alpha^2/r^2$ portion embodied in Φ at a radius of about 2×10^{-3} fermis. At short distances, the effective potential is highly attractive ($\sim -\alpha^2/r^2$) but not technically singular. Before going on to the solution of this equation for scattering states, we examine scattering solutions of the 1S_0 state.

The radial SLE is

$$\left\{ -\frac{d^2}{dr^2} - \frac{2\varepsilon_w\alpha}{r} - \frac{\alpha^2}{r^2} \right\} u = b^2(w)u = \frac{1}{4}(w^2 - 4m^2)u.$$

For scattering states, the Coulomb term and $-\alpha^2/r^2$ lead to the exact relativistic Coulomb wave functions

$$\begin{aligned} \bar{u} &= aF_\lambda(\eta, br) + cG_\lambda(\eta, br), \\ \lambda(\lambda + 1) &= -\alpha^2, \quad \lambda_\pm = \frac{1}{2}(-1 \pm \sqrt{1 - 4\alpha^2}) \\ \eta &= -\frac{\varepsilon_w\alpha}{b}. \end{aligned}$$

The lower sign correspond to peculiar solutions and the upper to the usual solutions. The asymptotic behavior of the regular Coulomb wave function is

$$F_{\lambda_\pm}(\eta, br \rightarrow \infty) \rightarrow \text{const} \times \sin(br - \eta \log 2br + \sigma_{\lambda_\pm} - \lambda_\pm \pi/2).$$

Two roots gives two sets of Coulomb phase shifts.

$$\begin{aligned} \delta_{\lambda_\pm} &= \sigma_{\lambda_\pm} - \lambda_\pm \pi/2, \\ \sigma_{\lambda_\pm} &= \eta \psi(\lambda_\pm + 1) + \sum_{n=0}^{\infty} \left(\frac{\eta}{\lambda_\pm + 1 + n} - \arctan\left(\frac{\eta}{\lambda_\pm + 1 + n}\right) \right) \end{aligned}$$

How might Eq. (8) lead to a resonance? The short distance behavior ($r \ll 2\alpha/w$) has the same usual and peculiar solutions as for the 1S_0 state. We solve for the phase shift by the

phase method of Calogero giving a nonlinear equation for the phase shift function. Starting with boundary conditions and integrating to infinity gives the phase shift. Built into the solutions are the Coulomb and negative barrier terms so that the equation is for the residual phase shift function due just to the real barrier and magnetic spin terms

$$\begin{aligned}\gamma'_{\pm}(r) &= -\frac{2}{b(r+2\alpha/w)^2}(\cos\gamma_{\pm}(r)F_{\lambda_{\pm}}(r) + \sin\gamma_{\pm}(r)G_{\lambda_{\pm}}(r))^2, \\ \gamma_{\pm}(0) &= 0.\end{aligned}$$

From this we obtain the total phase shift

$$\delta = \delta_1 + \sigma_1 = \gamma_{\pm}(\infty) + \sigma_{\lambda_{\pm}} + (1 - \lambda_{\pm})\pi/2.$$

This leads to no resonance for any energy for usual solution $\lambda_+ = \frac{1}{2}(-1 + \sqrt{1 - 4\alpha^2})$ and a 28 MeV resonance of 15 keV width for the peculiar solution $\lambda_- = \frac{1}{2}(-1 - \sqrt{1 - 4\alpha^2})$. The resonance disappears if the electron and positrons are not point particles.

4 Summary

The Two Body Dirac equations of constraint dynamics have dual origins in QFT and the classical relativistic two body problem. With the Adler-Piran potential the TBDE gives a very good fit to entire meson spectrum with just 3 invariant functions $A(r), V(r), S(r)$. We use the TBDE in the three two-body subsystems for baryon spectroscopy (we have not yet examined the three-body Dirac equations). The nonperturbative structure of the TBDE makes it more than competitive with other approaches since its QED applications reproduce numerically known perturbative spectrum. Finally, assuming point-like electron and positron, the TBDE predict new and peculiar positronium bound states and resonances.

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